

- $$\int x^3 e^{x^2} dx$$

$$\begin{aligned} \textcircled{1} \quad dy &= x e^{x^2} dx & u &= \frac{e^{x^2}}{2} \textcircled{1} \\ v &= x^2 & dv &= 2x dx \textcircled{0.5} \\ && &= \frac{x^2 \cdot x^2}{2} - \int x e^{x^2} dx \\ && &= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C \textcircled{0.5} \end{aligned}$$

2. (20 points) Study the convergence of these improper integrals without calculating them, then find their values when they converge:

a. $\int_1^{+\infty} \frac{1}{x^2(x^2+4)} dx$

(4) $\lim_{x \rightarrow +\infty} \frac{1}{x^2(x^2+4)} = 1 > 0$
 by L.C.T $\frac{1}{x^4}$
 then $\int_1^{+\infty} \frac{1}{x^4} dx$ & $\int_1^{+\infty} \frac{1}{x^2(x^2+4)} dx$ both converge and diverge
 p-integral
 $p=4 > 1$
 \Rightarrow converges $\Rightarrow \int_1^{+\infty} \frac{1}{x^2(x^2+4)} dx$ converges.

(3) Let $u = x^2$
 $\frac{1}{x^2(x^2+4)} = \frac{1}{u(u+4)}$
 $= \frac{A}{u} + \frac{B}{u+4}$
 $= \frac{1}{4u} - \frac{1}{4(u+4)}$
 $= \frac{1}{4x^2} - \frac{1}{4(x^2+4)}$
 $\int_1^{+\infty} \frac{1}{x^2(x^2+4)} dx = \int_1^{+\infty} \left(\frac{1}{4x^2} - \frac{1}{4(x^2+4)} \right) dx$
 $= \lim_{b \rightarrow +\infty} \int_1^b \left(\frac{1}{4x^2} - \frac{1}{4(x^2+4)} \right) dx$
 $= \lim_{b \rightarrow +\infty} \left[-\frac{1}{4x} - \frac{1}{4} \left(\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right) \right]_1^b$
 $= \lim_{b \rightarrow +\infty} \left[-\frac{1}{4b} - \frac{1}{8} \tan^{-1}\left(\frac{b}{2}\right) + \frac{1}{4} + \frac{1}{8} \tan^{-1}\left(\frac{1}{2}\right) \right]$
 $= -\frac{1}{8} \left(\frac{\pi}{2} \right) + \frac{1}{4} + \frac{1}{8} \tan^{-1}\left(\frac{1}{2}\right)$
 $= -\frac{\pi}{16} + \frac{1}{4} + \frac{1}{8} \tan^{-1}\left(\frac{1}{2}\right)$

b. $\int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x} + 1} dx$

even function

$$\int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x} + 1} dx = 2 \int_0^{+\infty} \frac{1}{e^x + e^x + 1} dx$$

$$\frac{1}{e^x + e^{-x} + 1} < \frac{1}{e^x} \quad \text{by D.C.T}$$

$$\int_0^{+\infty} \frac{1}{e^x} dx = \lim_{b \rightarrow +\infty} -e^{-x} \Big|_0^b = \lim_{b \rightarrow +\infty} -e^{-b} + 1 = 1$$

converges

by D.C.T $\int_0^{+\infty} \frac{1}{e^x + e^x + 1} dx$ converges

OK. L.C.T. $\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{e^x + e^{-x} + 1} dx$ converges.

$$\int_0^{+\infty} \frac{1}{e^x + e^{-x} + 1} dx = \int_0^{+\infty} \frac{e^x}{e^{2x} + e^x + 1} dx \quad u = e^x$$

$$= \lim_{b \rightarrow +\infty} \int_0^b \frac{e^x}{e^{2x} + e^x + 1} dx$$

$$\int_1^{e^b} \frac{du}{u^2 + u + 1} = \int_1^{e^b} \frac{du}{u^2 + u + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \int_1^{e^b} \frac{du}{(u + \frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{\frac{\sqrt{3}}{2}} \times \tan^{-1} \left(\frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2e^x + 1}{\sqrt{3}} \right) \Big|_0^b$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2e^b + 1}{\sqrt{3}} \right) - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \right)$$

$$\lim_{b \rightarrow +\infty} \left(\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2e^b + 1}{\sqrt{3}} \right) - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \right) = \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} \right) - \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} \right) = \frac{\pi}{\sqrt{3}} - \frac{2\pi}{3\sqrt{3}}$$

3. (6 points) Determine if the series $\sum_{n=1}^{+\infty} \frac{1}{(1+n^2) \tan^{-1}(n)}$ converges or not.

$$\sum_{n=1}^{+\infty} \frac{1}{(1+n^2) \tan^{-1}(n)}$$

let $f(x) = \frac{1}{(1+x^2) \tan^{-1}(x)}$

- f is \searrow because $x < y \Rightarrow f(x) > f(y)$
- f is cont. $\forall x$
- positive for $x \geq 0$
(\Rightarrow for $x \geq 1$)

\Rightarrow use integral test.

① $\int_1^{+\infty} f(x) dx$ and $\sum_{n=1}^{+\infty} f(n)$ both converge or diverge.

$$\int_1^{+\infty} \frac{1}{(1+x^2) \tan^{-1}(x)} dx = \lim_{b \rightarrow +\infty} \int_1^b \frac{1}{(1+x^2) \tan^{-1}(x)} dx$$

$$\begin{aligned} u &= \tan^{-1} x \\ du &= \frac{1}{1+x^2} dx \end{aligned}$$

$$= \lim_{b \rightarrow +\infty} \int \frac{du}{u}$$

$$= \lim_{b \rightarrow +\infty} \ln(\tan^{-1} x) \Big|_1^b$$

$$= \lim_{b \rightarrow +\infty} \ln(\tan^{-1} b) - \ln(\tan^{-1} 1)$$

$$= \ln\left(\frac{\pi}{2}\right) - \ln\left(\frac{\pi}{4}\right)$$

$$= \ln\left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}}\right)$$

$$= \ln 2 \Rightarrow \text{converges}$$

so $\sum_{n=1}^{+\infty} \frac{1}{(1+n^2) \tan^{-1}(n)}$ converges.

4. (20 points) Determine if each of the following series are absolutely convergent, conditionally convergent or divergent.

a. $\sum_{n=1}^{+\infty} \frac{(-1)^n n^n}{3^n n!}$

Ratio test for absolute convergence:

$$\lim_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow +\infty} \left(\frac{(n+1)^{n+1}}{3^{n+1} (n+1)!} \times \frac{3^n n!}{n^n} \right)$$

$$= \lim_{n \rightarrow +\infty} \left(\frac{1}{3} \frac{(n+1)^{n+1}}{(n+1) n^n} \right)$$

$$= \lim_{n \rightarrow +\infty} \left(\frac{(n+1)^n}{3 n^n} \right)$$

$$= \frac{1}{3} \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n$$

$$= \frac{1}{3} \cancel{\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n} e = \frac{e}{3} < 1$$

by ratio test $\sum_{n=1}^{+\infty} \frac{(-1)^n n^n}{3^n n!}$ is

absolutely convergent

\Rightarrow convergent.

not conditionally convergent

$$\begin{aligned}
 & 2 < n(n+1) < (n+1)(n+2) \\
 & \Rightarrow \sqrt{n(n+1)} < \sqrt{(n+1)(n+2)} \\
 & \Rightarrow \frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+2)}} \\
 & \Rightarrow u_n > u_{n+1}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} u_n = 0$$

$$u_n > 0$$

b. $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n(n+1)}}$

• Leibniz theorem \Rightarrow convergent

$$\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)}}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n(n+1)}} = 1 > 0$$

$$\sum_{n=1}^{+\infty} \frac{1}{n} \text{ harmonic series divergent}$$

by Limit Comparison test $\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)}}$ divergent

series is not absolutely convergent
it is conditionally convergent

c. $\sum_{n=2}^{+\infty} \frac{(-1)^n \sqrt{n}}{\ln(n)}$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n} = +\infty \quad (1)$$

\Rightarrow series not absolutely convergent by n^{th} -term test (1)

$$|a_n| \not\rightarrow 0 \Rightarrow a_n \not\rightarrow 0 \quad (2)$$

\Rightarrow series is divergent by n^{th} -term test. (1)

\Rightarrow not conditionally convergent

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n} = +\infty \neq 0 \Rightarrow \sum_{n=2}^{+\infty} \frac{\sqrt{n}}{\ln n} \text{ is divergent by n.t.t.}$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n \sqrt{n}}{\ln n} \text{ is divergent by n.t.t.}$$

5. (10 points) Find the interval of convergence and the radius of convergence of the power

series $\sum_{n=1}^{\infty} \frac{2^n}{n^2} (x-3)^n$

$$\lim_{n \rightarrow +\infty} \left| \frac{2^{n+1}}{(n+1)^2} (x-3)^{n+1} \cdot \frac{n^2}{2^n (x-3)^n} \right|$$

$$= \lim_{n \rightarrow +\infty} \left| \frac{2 n^2}{(n+1)^2} (x-3) \right|$$

$$= \lim_{n \rightarrow +\infty} 2|x-3| \cdot \frac{n^2}{(n+1)^2}$$

$$= 2|x-3| < 1$$

$$\Rightarrow |x-3| < \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} < x-3 < \frac{1}{2}$$

$$\Rightarrow +\frac{5}{2} < x < +\frac{7}{2}$$

series converges absolutely and converges
 $\forall \frac{5}{2} < x < \frac{7}{2}$

for $x < \frac{5}{2}$ & $x > \frac{7}{2}$

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow \rho > 1$$

$$|a_{n+1}| > |a_n| \quad \forall n > n_0$$

$$|a_n| \not\rightarrow 0$$

$$\Rightarrow a_n \not\rightarrow 0$$

by n^{th} term test
 series diverges if $x < \frac{5}{2}$ & $x > \frac{7}{2}$.

- for $x = \frac{5}{2}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ by Leibniz theorem
 series converges

- for $x = \frac{7}{2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2 > 1$
 converges

$$\boxed{R = \frac{1}{2} ; \text{I.O.C } \frac{5}{2} \leq x \leq \frac{7}{2}}$$

6. (9 points) Use Maclaurin series to estimate $\int_0^1 \frac{1 - \cos x}{x^2} dx$ with an error of magnitude less than 10^{-6} .

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$\textcircled{2} \quad 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots$$

$$\frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \dots$$

Term by term integration:

$$\textcircled{3} \quad \int_0^1 \frac{1 - \cos x}{x^2} dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \dots + \frac{(-1)^n x^{2n-2}}{(2n)!} + \dots \right) dx$$

$$= \frac{1}{2} x - \frac{x^3}{3(4!)} + \frac{x^5}{5 \times 6!} - \frac{x^7}{7 \times 8!} + \frac{x^9}{9 \times 10!} - \dots \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3 \times 4!} + \frac{1}{5 \times 6!} - \frac{1}{7 \times 8!} + \frac{1}{9 \times 10!} - \dots$$

$$2.7 \times 10^{-4} \quad 3.57 \times 10^{-6} \quad 3.66 \times 10^{-8} < 10^{-6}$$

$$\textcircled{2} \quad \int_0^1 \frac{1 - \cos x}{x^2} \approx \frac{1}{2} - \frac{1}{3 \times 4!} + \frac{1}{5 \times 6!} - \frac{1}{7 \times 8!}$$

$$\textcircled{1} \quad \text{with error} \leq \frac{1}{9 \times 10!} = 3.66 \times 10^{-8} < 10^{-6}$$

7. (10 points) Let $f(x, y) = \frac{y^4 - x^2 y^2}{x^4 + y^4}$

Determine if the limit of f as $(x, y) \rightarrow (0, 0)$ exists or not.

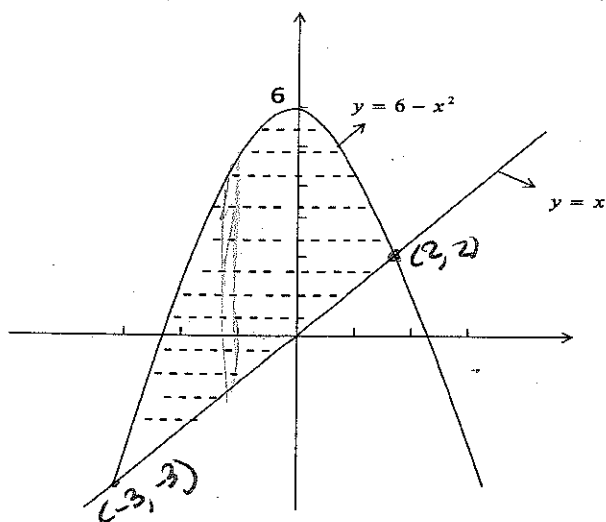
Let $x = ky$ or $y = kx$

$$\begin{aligned} \lim_{\substack{(x, y) \rightarrow (0, 0) \\ x = ky}} f(x, y) &= \lim_{y \rightarrow 0} \frac{y^4 - (ky)^2 y^2}{(ky)^4 + y^4} \\ &= \lim_{y \rightarrow 0} \frac{y^4 - k^2 y^4}{k^4 y^4 + y^4} \\ &= \lim_{y \rightarrow 0} \frac{y^4 (1 - k^2)}{y^4 (k^4 + 1)} \\ &= \frac{1 - k^2}{1 + k^4} \quad (\text{dependent of } k) \end{aligned}$$

by 2 - path test

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \quad \text{does not exist}$$

8. (10 points) Find the volume under the parabolic cylinder $z = x^2$ whose base is the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy -plane.



④
$$\int_{-3}^2 \int_x^{6-x^2} x^2 dy dx$$

①
$$= \int_{-3}^2 (x^2 y) \Big|_x^{6-x^2} dx$$

①
$$= \int_{-3}^2 x^2 (6 - x^2 - x) dx$$

①
$$= \int_{-3}^2 (6x^2 - x^4 - x^3) dx$$

①
$$= \left[\frac{6x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4} \right]_{-3}^2$$

①
$$= 2(2^3 + 3^3) - \frac{1}{5}(2^5 + 3^5) - \frac{1}{4}(2^4 - 3^4)$$

$$= \underline{31.25}$$

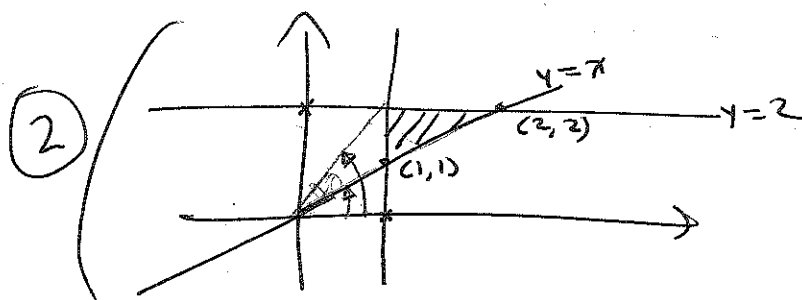
7 pts

①
$$\begin{aligned} y &= y \\ 6 - x^2 &= x \\ 6 - x^2 - x &= 0 \\ x &= 2, x = -3 \end{aligned}$$

9. (8 points) Find the double integral $I = \int_1^2 \int_x^2 \frac{1}{x \ln y} dy dx$

$$y: x \rightarrow 2$$

$$x: 1 \rightarrow 2$$



$$I = \int_1^2 \int_x^2 \frac{1}{x \ln y} dy dx$$

② $\left(= \int_1^2 \int_1^y \frac{1}{x \ln y} dx dy \right)$ ① reverse

② $\left(= \int_1^2 \left(\frac{1}{\ln y} \ln x \right) dy \right)$

$= \int_1^2 \left(\frac{1}{\ln y} (\ln y) \right) dy$

① $= \int_1^2 dy = y \Big|_1^2 = 2 - 1 = 1$

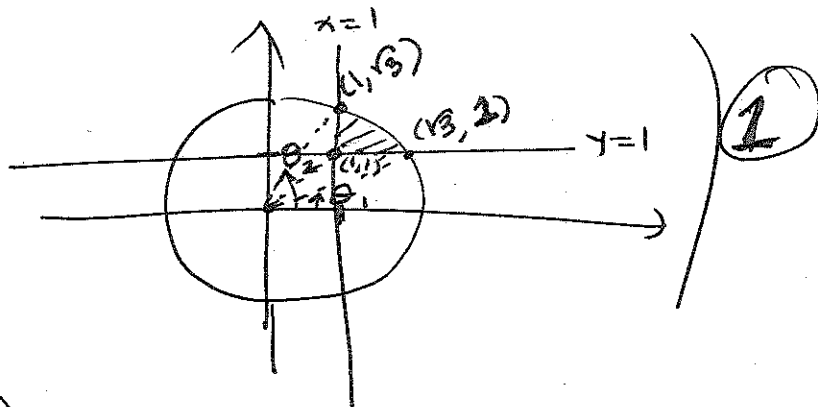
10. (12 points) Find the double integral $I = \iint_R \frac{1}{\sqrt{x^2 + y^2}} dy dx$ where R is the region

inside the circle $x^2 + y^2 = 4$ satisfying $x \geq 1$ and $y \geq 1$

$$\begin{aligned} x &= 1 \\ r \cos \theta &= 1 \\ r &= \frac{1}{\cos \theta} = \sec \theta \end{aligned} \quad (1)$$

$$\begin{aligned} y &= 1 \\ r \sin \theta &= 1 \\ r &= \frac{1}{\sin \theta} = \csc \theta \end{aligned} \quad (1)$$

$$\begin{aligned} \tan \theta_1 &= \frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \frac{\pi}{6} \\ \tan \theta_2 &= \frac{\sqrt{3}}{1} \Rightarrow \theta_2 = \frac{\pi}{3} \end{aligned} \quad (1)$$



$$\begin{aligned} y &= 1 \\ x^2 + y^2 &= 4 \end{aligned}$$

$$\begin{aligned} x^2 &= 3 \\ x &= \pm \sqrt{3} \end{aligned}$$

$$x = 1$$

$$\begin{aligned} y^2 &= 3 \\ y &= \pm \sqrt{3} \end{aligned}$$

$$(4) \int_{\pi/6}^{\pi/4} \int_{\csc \theta}^2 \frac{1}{r} r dr d\theta$$

$$+ \int_{\pi/4}^{\pi/3} \int_{\sec \theta}^2 \frac{1}{r} r dr d\theta$$

$$\begin{aligned} (1) &= \int_{\pi/6}^{\pi/4} (r^2 \Big|_{\csc \theta}^2) d\theta + \int_{\pi/4}^{\pi/3} (r^2 \Big|_{\sec \theta}^2) d\theta \\ &= \int_{\pi/6}^{\pi/4} (2 - \csc^2 \theta) d\theta + \int_{\pi/4}^{\pi/3} (2 - \sec^2 \theta) d\theta \end{aligned}$$

Good Luck ☺

$$(1) = 2\theta + \ln |\csc \theta + \cot \theta| \Big|_{\pi/6}^{\pi/4} + 2\theta - \ln |\sec \theta + \tan \theta| \Big|_{\pi/4}^{\pi/3}$$

$$(1) = \frac{\pi}{3} - 2 \ln(2 + \sqrt{3}) + 2 \ln(\sqrt{2} + 1)$$